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Quantization of non-polynomial field theories

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Abstract: We re-examine the quantization of a class of non-polynomial scalar field theories which interpolates continuously from a free one to ϕ^4 theory. The quantization of such theories is problematic because the Feynman rules may not be directly obtained. We give a means for calculating the correlation functions in this theory. The Feynman rules developed here shall enable further progress in the understanding of the triviality of ϕ^4 theory in four dimensions.

Keywords: Renormalization Regularization and Renormalons, Effective Theories, Nonperturbative Effects.

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Contents

1 Introduction

In previous years, a novel perturbative scheme was found enabling one to calculate correlation functions in certain non-polynomial scalar field theories. The interaction is taken to be $(\Phi^2)^{2-\delta}$ [1, 2, 3], and gives a non-polynomial theory containing logarithms,

$$\begin{aligned} (\Phi^2)^{2-\delta} &= \phi^4 e^{-\delta \ln \phi^2} \\ &= \phi^4 - \delta\phi^4 \ln \phi^2 + \ldots \ . \end{aligned} \tag{1.1}$$

Such a theory continuously interpolates between a free theory and the standard ϕ^4 scalar model.

There are a few interesting aspects of a theory with this potential. Perhaps the most intriguing one pertains to the fact that ϕ^4 theory in four dimensions may actually be non-interacting, although no analytical evidence for this has been found. This is the so-called triviality problem in four dimensions which to this day still remains unsettled, despite strong numerical evidence. To ask whether a ϕ^4 theory remains trivially interacting when coupled to other fields, such as a non-abelian gauge theory, is a different question. However, in light of the fact that the Higgs mechanism is the foundation of the electroweak sector it is an important question to address.

Varying the parameter δ in the theory with the interaction above in eq. (1.1) allows one to interpolate from $\delta = 1$ along a path $\phi^4|\phi|^{-2\delta}$ and explore the continuous relationship between the dimension of space-time d and the interaction parameter δ. The scalar field theory with a potential ϕ^4 is expected to be interacting in $d < 4$, where the coupling obtains a positive mass scale. Intuitively, we expect to find non-trivial interactions when the coupling constant has a positive mass scale, so that the behavior of the full field theory should be tied strongly to both d and δ. Indeed, the triviality question can be phrased in the context of whether or not the ϕ^4 operator in the full quantum field theory may acquire scaling dimensions away from four, which effectively generates a scale for

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the coupling. The analysis of Greens functions in the space of values (d, δ) sheds further insight into the triviality problem.

Further issues addressed in the study of this theory concerns the behavior of field theories containing an infinite number of monomial interactions. Within the analysis of the renormalization group we know that to any finite order in perturbation theory, only a few operators, namely the relevant ones, dominate the behavior of the Greens functions at low energy. Corrections arising from higher-dimensional interactions, and which are suppressed by the cutoff used to regularize the theory, are suppressed by powers of the renormalization scale. In the case of scalar theories in four dimensions the mass operator ϕ^2 is the only relevant one, while the interaction ϕ^4 with classically zero scaling dimensions may or may not be: it is a marginal operator which perturbatively acquires non-vanishing scaling dimensions via quantum effects. As we will see, the theory with an interaction above may be treated as a theory with an infinite number of interactions simulating the non-polynomial potential. Taking into account the effects of the infinite tower of couplings remarkably changes the character of the Greens functions. The issues of small perturbations of the theory, and the combined effect of a tower of irrelevant interactions, is addressed in the context of the non-polynomial theory.

The first difficulty in working with such a theory is how to formulate a perturbative expansion of the Greens functions; the potential is not differentiable at $\phi = 0$ and hence does not admit a Taylor expansion. There are no textbook Feynman rules present. One may resort to letting the mass parameter m^2 become negative, followed by an expansion about the broken phase from which the scalar field acquires a non-vanishing vacuum value. Another option, which we develop in these notes, is to find an analytic continuation in δ in which the Greens functions may be defined. We will see that there are several different means of defining this theory, all of which lead to the same perturbative results.

This work is broken into several sections. In section 2, we develop the perturbative means for calculating the Greens functions. In section 3 we perform the next-to-lowest order (in λ) calculation of the four-point function and investigate the behavior as a function of the dimension and interaction parameter.

2 Perturbative expansion

We take our theory to be a free massive scalar theory coupled to an interaction of the form

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{4}(\Phi^2)^{2-\delta} \ , \tag{2.1}$$

and are interested both in exploring the behavior of the theory as a function of the dimension d, as well as in the possible momentum cutoff Λ. In d dimensions, the mass scales of the fields and coupling constants are

$$[\phi] = \frac{d}{2} - 1$$

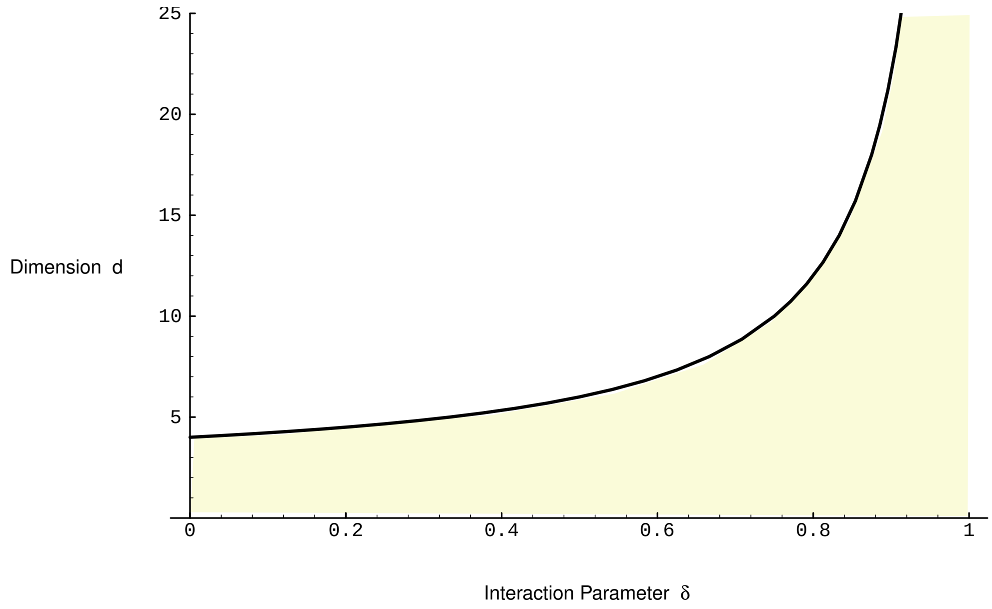

Figure 1: The shaded region denotes the parameter space of (d, δ) leading to naive power-counting renormalizability.

$$[\lambda] = 2\delta - (d-4)(1-\delta) \ , \tag{2.2}$$

and the field theory is naively power counting renormalizable for

$$d < 4 + \frac{2\delta}{1-\delta} \ . \tag{2.3}$$

Note that even in four dimensions the coupling has a mass scale when δ is not zero. The space of naive power-counting renormalizable theories, in which $[\lambda] \geq 0$, is illustrated in fig. 1 as a function of the dimension and the interaction parameter δ.

For general values of d and δ the coupling constant λ becomes a classical mass scale. We extract the arbitrary scale μ to keep the coupling constant λ dimensionless; the full Lagrangian in $d = 4 - \epsilon$ dimensions is then

$$\mathcal{L}_d = \frac{1}{2}\partial\phi\partial\phi - \frac{1}{2}m^2\phi^2 - \mu^{2\delta+\epsilon(1-\delta)}\frac{\lambda}{4!}(\Phi^2)^{2-\delta}. \tag{2.4}$$

We will drop for now the μ scale and re-insert it later.

In the second formulation we define the theory in Euclidean space with a momentum cutoff Λ, thus explicitly breaking Poincare invariance. With this regularization the dimension of the coupling constant by the criteria of naturalness [7] should be replaced by the appropriate power of the cutoff Λ. The Lagrangian becomes

$$\mathcal{L}_\Lambda = \frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}m^2\phi^2 + \Lambda^{2\delta}\frac{\lambda}{4!}(\Phi^2)^{2-\delta}. \tag{2.5}$$

Again we drop the $\Lambda^{2\delta}$ for now and insert it back later. In the more general case we may give the dimensions $(\Lambda^{2-a}m^a)^\delta$ to the coupling constant (where $2 \geq a \geq 0$). The details of deriving the effective Feynman rules do not depend on the two choices of the regularization schemes.

2.1 Perturbative rules

In this section we develop the formalism for doing perturbative calculations despite the non-Taylor expandable interaction in (1.1). In the presence of a source term $\mathcal{L} = \int J\phi$, the connected Greens functions are found by the appropriate functional derivatives of $Z[J]$ with respect to the source $J(x_i)$. The path integral is defined as

$$Z[J] = \int [d\Phi] \exp\Big(i\langle \frac{1}{2}\partial_\alpha \Phi \partial^\alpha \Phi - \frac{1}{2}m^2\Phi^2 - \frac{\lambda}{4!}(\Phi^2)^{2-\delta} + J\Phi\rangle\Big) , \tag{2.6}$$

from which we derive the Greens functions

$$\begin{aligned} G^{(n)}(x_1, x_2, \ldots, x_n) &= (-i)^n \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} \ln Z[J]\,|_{J=0} \\ &= \langle \phi(x_1) \cdots \phi(x_n) \rangle . \end{aligned} \tag{2.7}$$

For polynomial field theories the standard means of generating the connected Greens functions is derived by explicitly taking the functional derivatives of $\ln Z[J]$ and applying Wicks theorem in the expansion of the functional integral. However, the potential in question is not polynomial and there is no analog of Wick's theorem that disentangles a non-integer number of fields. Clearly then there are no Feynman rules in the standard sense for this theory.

The first course of action we take is the following: We perform an expansion of the potential in terms of Laguerre polynomials (given by a sum of monomials) followed by a resummation of all of the self-energy corrections. In effect this converts the original non-polynomial potential into an infinite sum of interactions, all of which are normal-ordered by construction.

The generalized Laguerre polynomials [5] are

$$L_n^\alpha(x) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!(n-k)!} . \tag{2.8}$$

Note that we can replace the upper limit with ∞ since the argument of the sum vanishes for $k > n$. The above polynomials satisfy the generalized Laguerre differential equation. Next we use an expansion which is uniformly convergent for all $\phi^2 > 0$ and $-\delta > -\frac{1}{2}(\alpha + 1)$:

$$(\phi^2)^{-\delta} = \Gamma(1+\alpha-\delta)\Gamma(1-\delta) \sum_{n=0}^{\infty} \frac{(-)^n L_n^\alpha(\phi^2)}{\Gamma(1+\alpha+n)\Gamma(1-\delta-n)} . \tag{2.9}$$

After inserting the Laguerre expansion into (2.9) and using the Gamma function reflection identity we obtain the double infinite sum representing the interaction

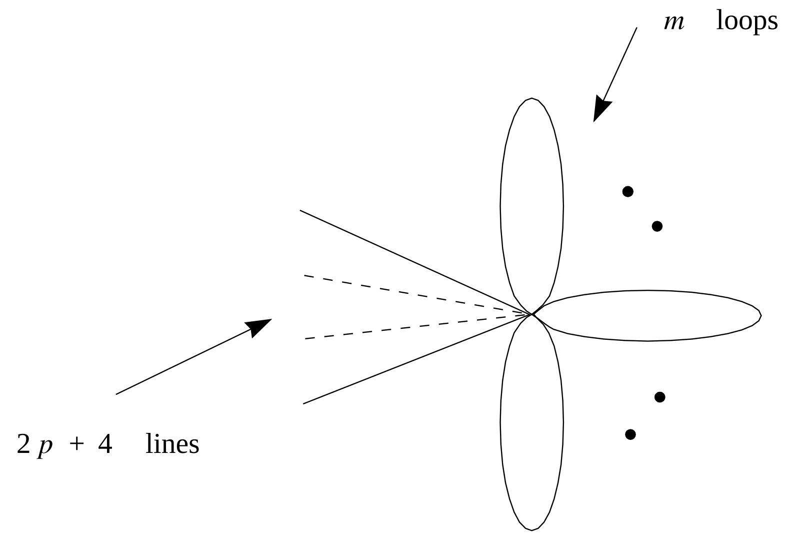

Figure 2: An effective $2p+4$-point vertex is found by self-contracting $2m$ lines into tadpoles from the $2m+2p+4$-point vertex $\lambda_{2p+2m+4}$.

$$(\phi^2)^{-\delta} = -\Gamma(1+\alpha-\delta)\Gamma(1-\delta)(\frac{\sin\pi\delta}{\pi})\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(n+\delta)}{\Gamma(1+\alpha+k)}\frac{(-\phi^2)^k}{k!(n-k)!} . \tag{2.10}$$

At this point we have more or less formally manipulated the non-polynomial interaction and rewritten it as an infinite sum of polynomials. It is important to note that this (non-Taylor) expansion is uniformly convergent in the range $\phi^2 > 0$.

Using the expansion (2.10) we can read off the Feynman rules. Denote by λ_{2k+4} the value of the $2k$-point tree vertex. Then the potential is

$$V(\phi) = \sum_{k=0}^{\infty}\lambda_{2k+4}\frac{\phi^{2k+4}}{(2k+4)!} , \tag{2.11}$$

where

$$\lambda_{2k+4} = \lambda_\alpha(-)^{k+1}\frac{\Gamma(2k+5)}{\Gamma(\alpha+k+1)\Gamma(k+1)}\sum_{n=0}^{\infty}\frac{\Gamma(n+\delta)}{\Gamma(1+n-k)} , \tag{2.12}$$

and

$$\lambda_\alpha = \lambda\ \Gamma(1+\alpha-\delta)\Gamma(1-\delta)(\frac{\sin\pi\delta}{\pi}) . \tag{2.13}$$

The extra factor of $\Gamma(2k+5)$ has been divided out of the definition of the coupling λ_{2k+4} to make up for the combinatorical factor associated with Wick's theorem. However, the couplings λ_{2k+4} do not make sense because the sums in (2.13) do not converge; upon summing over all self-interactions, illustrated in fig. 2, we shall derive the full vertices. The resummation effectively normal-orders all of the polynomial interactions in (2.12).

If we sum over all of the contributions we arrive at new zeroth-order couplings g_{2p} which by definition correspond to a normal-ordered (::) potential

$$V(\phi) = \sum_{p=0}^{\infty} g_{2p} \frac{:\phi^{2p}:}{(2p)!} \ . \tag{2.14}$$

We compute the sum over all daisy diagrams by taking the vertices in eq. (2.12) and summing over all graphs formed by connecting lines into tad-pole configurations. In other words, a naive vertex with $2m+2p$ lines can have $2m$ of its lines contracted, thus forming m tadpoles, leading to an effective vertex with $2p$ external lines. The graphs have a symmetry factor $\frac{1}{2^m m!}$. Summing over all the tadpole contractions of $2m$ lines from the vertices with $2m+2p$ lines gives an infinite sum expression for the effective vertex (where here $p \geq -2$) ,

$$g_{2p+4} = \sum_{m=0}^{\infty} \lambda_{2p+2m+4} (\frac{I}{2})^m \frac{1}{m!} \ . \tag{2.15}$$

The integral I is defined to be the tadpole

$$I \equiv \int \frac{d^d l}{(2\pi)^d} \frac{i}{l^2 - m^2} = \Gamma(1 - \frac{d}{2})(m^2)^{\frac{d}{2}-1}(4\pi)^{-\frac{d}{2}} \ . \tag{2.16}$$

Adding together all of the contributions illustrated in fig. 2 then gives the infinite sum expression which describes the new coupling g_{2p+4}

$$\begin{aligned} g_{2p+4} = \ & \lambda_\alpha \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(2p+2k+5)}{\Gamma(p+k+\alpha+1)\Gamma(p+k+1)} \times \\ & \times \frac{\Gamma(n+\delta)}{\Gamma(k+1)\Gamma(n-p-k+1)} (\frac{I}{2})^k (-)^{p+k} \ . \end{aligned} \tag{2.17}$$

The evaluation of the sums in (2.17) may be performed as follows: first we shift k by $k \to k-p$ since the inverse gamma function contributes zero for $k < p$. Using the identity

$$\frac{\Gamma(2k+5)}{\Gamma(k-p+3)} = 2^{2k+4} \pi^{-\frac{1}{2}} (\frac{\partial}{\partial \beta})^p \beta^{k+2} \int_{-\infty}^{\infty} dt e^{-t^2} (t^2)^{k+2} |_{\beta=1} \ , \tag{2.18}$$

we may rewrite the effective coupling in eq. (2.18) as

$$\begin{aligned} g_{2p} \ = \ & 16 \lambda_\alpha \ \pi^{-\frac{1}{2}} (\frac{I}{2})^{2-p} (\frac{\partial}{\partial \beta})^p \beta^2 \int_{-\infty}^{\infty} dt \ t^4 e^{-t^2} \times \\ & \times \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(n+\delta)}{\Gamma(k+1+\alpha)} \frac{1}{\Gamma(k+1)\Gamma(n-k+1)} (-2I\beta t^2)^k |_{\beta=1} \ . \end{aligned} \tag{2.19}$$

We have interchanged the order of the sum and integration, which is similar to performing an analytic continuation in the expression in the parameter δ. Performing the summation gives the compact result,

$$g_{2p} = 16 \lambda_\alpha \ \pi^{-\frac{1}{2}} (\frac{I}{2})^{2-p} (\frac{\partial}{\partial \beta})^p \beta^2 \int_{-\infty}^{\infty} dt \ t^4 e^{-t^2} \lambda (2\beta t^2 I)^{-\delta} \ . \tag{2.20}$$

As a final step, the integral over t and the derivatives with respect to β are completely split and may be trivially evaluated: we arrive at the result

$$g_{2p} = \lambda_\alpha \frac{\Gamma(5-2\delta)}{\Gamma(3-\delta-p)} (\frac{I}{2})^{2-p-\delta} \qquad p \geq 0 \quad , \tag{2.21}$$

whereby in the derivation of the couplings g_{2p} in (2.21) we have effectively normal ordered the operators ϕ^{2p}. The initial Lagrangian with the interaction $(\Phi^2)^{2-\delta}$ has been effectively redefined as an infinite sum of polynomial interactions

$$L = \frac{1}{2}\partial\phi\partial\phi - \frac{1}{2}m^2\phi^2 - \sum_{p=0}^{\infty} \frac{g_{2p}}{(2p)!} : \phi^{2p} : \quad . \tag{2.22}$$

Lastly, the effective theory above is constructed to *all orders* in the free parameter δ. One can check that the usual free theory and ϕ^4 theory are obtained in the limits $\delta = 1$ and 0, respectively. Furthermore, at tree-level all of the interactions in eq. (2.22) at $2n \geq 4$-point vanish due to the suppression factor $I^{2-p-\delta}$ within the couplings g_{2p}.

One of the main motivations for the study of this theory, besides being a means for dealing with scalar effective actions containing logarithms, was to study the triviality problem in four dimensions. In future work we shall present various loop calculations and analytical evidence [9].

3 Conclusions

In this work we have re-examined the quantization of a class of field theories with an interaction of the form $(\phi^2)^{2-2/\delta}$. Although the theory does not admit the usual perturbative definition of the Greens functions, we have mapped the interaction to a scalar theory containing an infinite number of polynomial terms. The calculation of Greens functions then follow as usual from the perturbative rules. In subsequent work we shall use our prescription for defining the non-polynomial theory to give some of the first analytical evidence supporting the triviality of ϕ^4 theory in four dimensions.

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